Comparison of Experimental Ultrahigh Energy Cosmic Ray Spectra using the Bayes Factor

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***Abstract*. A frequent problem in ultrahigh energy cosmic ray physics is the comparison of energy spectra. Given two or more measurements of the energy spectrum, the question is whether the spectra agree within their statistical and systematic uncertainties. We develop a method to directly compare two different energy spectra for ultrahigh enery cosmic rays that takes into account systematic errors and does not require the introduction of theoretical models for the shape of the energy spectrum. The consistency between the observations is expressed in terms of a Bayes Factor, a standard statistic defined as the ratio of a separate parent source hypothesis to a single parent source hypothesis. The method can be used to compare enery spectra measured by different experiments, or when comparing the energy spectra for different regions of the sky.**

***Keywords*: ultra high-energy cosmic rays, energy spectrum, bayes factor**

I. Introduction

The energy spectrum of ultrahigh energy cosmic rays is sensitive to the production and transport mechanisms of ultrahigh energy particles, as well as cosmic ray mass composition. Therefore, the spectrum is one of the most important quantities available to experiment. However, the low cosmic ray flux at earth requires indirect detection of ultrahigh energy particles via observations of extensive air showers. Such measurements are plagued by $20\%-30\%$ systematic uncertainties in the energy determination, as well as very poor high-energy statistics. Even in large-aperture experiments, the rapid decrease of the flux with energy means that the highest energy values of the differential spectrum are determined by several events (or less).

The large statistical and systematic uncertainties typical of observations of the ultrahigh energy spectrum complicate comparisons between experimental measurements and between measurements and model predictions. Due to the complexities of the calculation, such comparisons often contain a rigorous treatment of statistical uncertainties while ignoring or misusing systematic effects. For example, it is common practice to simply shift experimental spectra by their 1σ systematic energy uncertainties and then compare the shifted fluxes using the statistical uncertainties. Such an analysis fails to account for the probability of a systematic shift in energy scale applied to the data.

To handle the comparison of energy spectra between experiments and between measured and modeled data, we describe a technique that naturally incorporates the statistical and systematic uncertainties of experimental measurements. The technique is based upon the Bayes Factor $\mathcal{R}$, a likelihood ratio commonly used in Bayesian hypothesis testing. In our case, the hypotheses of interest are the probability that two (or more) experimental spectra are derived from a single "parent" spectrum, versus the probability that the measurements are consistent with two (or more) parent distributions. The latter hypothesis is motivated, for example, by the possibility of different production and propagation mechanisms and/or particle composition in different regions of the sky.

We describe the Bayes Factor technique in Section II. An application to experimental data will be given in Section III.

II. Bayes Factor

The calculation of the Bayes Factor $\mathcal{R}$ is a variant of the method given in [1] to compare two Poisson parameters. Given binned fluxes $\vec{\mathcal{F}_1}$ $=\{\mathcal{F}_{1,i}\}$ and $\vec{\mathcal{F}_2}$ $=\{\mathcal{F}_{2,i}\}$ observed in two different experiments, $\mathcal{R}$ is the probability ratio that the measurements arise from two parent distributions vs. one parent distribution:

$$\mathcal{R} = \frac{P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2 | \mathcal{H}_{\text{two-parent}})}{P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2 | \mathcal{H}_{\text{one-parent}})}. \tag{1}$$

Because the probabilities in this ratio are Bayesian likelihoods, we do not make prior assumptions about the relative probabilities of the one- and two-parent hypotheses.

We compare the observed fluxes with the expected values $\vec{f_1}$ $=\{f_{1,i}\}$ and $\vec{f_2}$ $=\{f_{2,i}\}$ given a particular hypothesis. It is convenient to express the expectation in terms of the total expected counts from both experiments $\{\eta_i\}$ and a set of experimental weights $\{w_i\}$, such that

$$f_{1,i} = w_i \eta_i, \qquad f_{2,i} = (1 - w_i)\eta_i. \tag{2}$$

In the single-parent hypothesis, the weights are simply the relative exposure of the two experiments:

$$w_i = \frac{(\text{Exposure 1})_i}{(\text{Exposure 1})_i + (\text{Exposure 2})_i}. \tag{3}$$

In the two-parent hypothesis, the weights are allowed to take on any value between 0 and 1.

A. Expanding the Likelihood Ratio

Since the expected fluxes $\vec{f_1}$ and $\vec{f_2}$ are unknown, we marginalize these parameters in the numerator and denominator of the Bayes Factor, so that eq. (1) becomes

$$\mathcal{R} = \frac{\iint P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{f}_1, \vec{f}_2)\ P(\vec{f}_1, \vec{f}_2|\mathcal{H}_{\text{two}})\ d\vec{f}_1\ d\vec{f}_2}{\iint P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{f}_1, \vec{f}_2)\ P(\vec{f}_1, \vec{f}_2|\mathcal{H}_{\text{one}})\ d\vec{f}_1\ d\vec{f}_2} = \frac{\iint P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{w'}, \vec{\eta})\ P(\vec{w'}, \vec{\eta})\ d\vec{w'}\ d\vec{\eta}}{\int P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{w}, \vec{\eta})\ P(\vec{w}, \vec{\eta})\ d\vec{\eta}}. \quad (4)$$

In the denominator, the marginalization over the experimental weights collapses since we equate the weights with the relative experimental exposures; in the numerator, every possible relative exposure is allowed since the experiments could be observing two different fluxes. Prior model restrictions on the weights and expected counts can be introduced via the probabilities $P(\vec{w}, \vec{\eta})$.

The expected and observed counts are related by the quantity $P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{w}, \vec{\eta})$, which describes the probability of observing counts $\{\mathcal{F}_{1,i}\}$ and $\{\mathcal{F}_{2,i}\}$ given expected counts $\{w_i\eta_i\}$ and $\{(1-w_i)\eta_i\}$. This probability factorizes for the two experiments. We account for the statistical uncertainties in the flux counts by parameterizing the probability as the product of two Poisson distributions:

$$P(\mathcal{F}_1, \mathcal{F}_2|w, \eta)_i = \frac{(w_i\eta_i)^{\mathcal{F}_{1,i}}\ e^{-w_i\eta_i}}{\mathcal{F}_{1,i}!} \frac{((1-w_i)\eta_i)^{\mathcal{F}_{2,i}}\ e^{-(1-w_i)\eta_i}}{\mathcal{F}_{2,i}!} \quad (5)$$

Assuming flat prior probabilities for the weights and expected counts, eq. (4) then reduces to

$$\mathcal{R} = \frac{\prod_{i=1}^{N} \frac{1}{\mathcal{F}_{1,i}+\mathcal{F}_{2,i}+1}}{\prod_{i=1}^{N} w_i^{\mathcal{F}_{1,i}} (1-w_i)^{\mathcal{F}_{2,i}} \frac{(\mathcal{F}_{1,i}+\mathcal{F}_{2,i})!}{\mathcal{F}_{1,i}!\mathcal{F}_{2,i}!}} \quad (6)$$

B. Introducing Systematic Uncertainties

To account for experimental uncertainties in the energy scale, we introduce parameters s_1 and s_2 that represent deviations of the mean energy scales of experiments 1 and 2 from the true energy scale. Since the deviations from the true energies are not known, we marginalize s_1 and s_2:

$$\mathcal{R} = \frac{\iiiint P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{w'}, \vec{\eta}, s_1, s_2)}{\iiint P(\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2|\vec{w}, \vec{\eta}, s_1, s_2)} \times \frac{P(\vec{w'}, \vec{\eta}, s_1, s_2)\ d\vec{w'}\ d\vec{\eta}\ ds_1\ ds_2}{P(\vec{w}, \vec{\eta}, s_1, s_2)\ d\vec{\eta}\ ds_1\ ds_2} \quad (7)$$

It is convenient to parameterize the probability distributions of the scale shifts as zero-mean Gaussians with widths corresponding to the systematic energy uncertainties σ_1 and σ_2 of each experiment. Including this energy smearing, the Bayes Factor becomes

$$\mathcal{R} = \frac{\iint \left[\prod_{i=1}^{N} \frac{1}{\mathcal{F}_{1,i}+\mathcal{F}_{2,i}+1}\right]}{\iint \left[\prod_{i=1}^{N} w_i^{\mathcal{F}_{1,i}} (1-w_i)^{\mathcal{F}_{2,i}} \frac{(\mathcal{F}_{1,i}+\mathcal{F}_{2,i})!}{\mathcal{F}_{1,i}!\mathcal{F}_{2,i}!}\right]} \times \frac{\mathcal{N}(s_1;0,\sigma_1)\ \mathcal{N}(s_2;0,\sigma_2)\ ds_1\ ds_2}{\mathcal{N}(s_1;0,\sigma_1)\ \mathcal{N}(s_2;0,\sigma_2)\ ds_1\ ds_2} \quad (8)$$

Note that when the experimental energies are smeared, the flux values in each bin also change. Hence, the fluxes are functions of s_1 and s_2 and must be incorporated into the integral over the detector resolutions. This imposes some practical limitations on the calculation of eq. (8). For example, the energy smearing must occur in discrete steps which are small relative to the full energy range of interest, but large enough to limit the total number of computations. One must also cut off the size of the shifts; values up to $\pm 5\sigma$ from the mean energies, weighted by the Gaussian energy uncertainties, provide excellent accuracy. A detailed discussion of the implementation of this algorithm is provided in [2].

III. Application

We have applied this calculation to recently published results from the HiRes and Auger Collaborations, using data up to August 2008. Results of the calculation will be presented at the conference.

References

[1] H. Jeffreys. *Theory of Probability*. Oxford, 1939.
[2] B. M. Connolly et al. *Phys. Rev.*, D74:043001, 2006.