On a Stochastic Approach to Cosmic-ray Modulation

Chunsheng Pei*, **John W. Bieber***, **R. Adri Burger**†, **John Clem*** and **William H. Matthaeus***

*Bartol Research Institute, Department of Physics and Astronomy, University of Delaware, Newark, Delaware, 19716
†Unit for Space Physics, Private Bag X6001, North-West University, Potchefstroom Campus, South Africa

***Abstract*. We present detailed numerical solutions of the steady state modulation of galactic cosmic rays for a three-dimensional model with or without drifts by two distinct approaches, the stochastic (Monte-Carlo) method and the finite difference method. We derive the drift velocity formula based on a planar current sheet. We calculate the energy spectra by using the same Parker magnetic field, same nominal interplanetary transport parameters, and same boundary conditions for both methods. We used diffusion coefficients $\kappa_\parallel = 10\kappa_\perp = 2 \times 10^{23}\beta\mathbf{P}$ for most runs, where $\kappa_\parallel$ is the parallel diffusion coefficient in units of cm^{2}s^{-1}, $\kappa_\perp$ is the perpendicular diffusion coefficient, P is particle rigidity in GV, and β is the ratio of particle speed and the speed of light. We assume the heliospheric radius is 100 AU and neglect shocks and other acceleration mechanisms inside the heliosphere. Although other stochastic methods use Cartesian coordinates, we demonstrate that we can use spherical coordinates for Parker's spiral field, which makes the code much more efficient by reducing coordinate transformations at each step. We show that excellent agreement exists between solutions obtained by these two different methods. We believe this work constitutes the most thorough benchmarking of the two methods performed to date.**

***Keywords*: Stochastic approach, cosmic-ray modulation, numerical method.**

I. Introduction

The solar modulation of galactic cosmic rays is described by the Fokker-Planck equation (FPE) [15] and [9],

$$\frac{\partial f}{\partial t} = \nabla\cdot(\kappa\cdot\nabla f) - \nabla\cdot(Vf) + \frac{1}{3}(\nabla\cdot V)p\frac{\partial f}{\partial p} + Q\ . \quad (1)$$

Here $f(p, x_i, t)$ is the omni-directional distribution function, x_i is the spatial variable, p is the momentum magnitude, V_i is the solar wind velocity, and Q is the source term. The spatial diffusion tensor, κ_{ij}, can be decomposed into two parts, $\boldsymbol{\kappa}_s$, the symmetric part, and $\boldsymbol{\kappa}_A$, the anti-symmetric part. The divergence of the anti-symmetric part is the drift velocity, V_d. The symmetric tensor, $\boldsymbol{\kappa}_s$, represents the spatial diffusion parallel $\kappa_\parallel$ and perpendicular $\kappa_\perp$ to $\mathbf{B}$. We will neglect Q in this paper.

Two major numerical methods have been developed to solve Equation (1). One is the finite difference method which has been used by [7], [10]. For the current paper, a steady-state three-dimensional version [3], following the approach of [12], is used. The other one is the stochastic method or Monte Carlo method used by [8], [11], [14], [1], [18], [6], [17]. For 1D and 2D problems, the finite difference method is much faster than the stochastic method. However, for 3D configuration and for a complex interplanetary magnetic field, for example, if we consider the wavy current sheet and the time dependent problem, the stochastic method outperforms the finite difference method. For years, the results of these two methods have not been throughly cross-examined. In this paper we provide a direct comparison between the results of these two methods for different scenarios.

It is generally believed that for 3D stochastic method the integration must be performed in a Euclidean space to apply Ito's formula [18]. We show that for a special interplanetary magnetic field, the generally accepted Parker field, we can actually use the spherical coordinates avoiding the transformation between the two coordinates. This simplification saves a great deal of computation time.

II. 2D Stochastic Model

Let us start with 2D stochastic method without considering drift and cross terms of κ which has been extensively used in literature. The 2D transport of cosmic rays in the heliosphere can be re-written according to Equation (1),

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_{rr}\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\kappa_{\theta\theta}\sin\theta\frac{\partial f}{\partial\theta}\right)\\ &+ \frac{1}{3r^2}\frac{\partial}{\partial r}\left(r^2V\right)\frac{\partial}{\partial T}\left(\Gamma Tf\right) - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2Vf\right)\ , \quad (2)\end{aligned}$$

with (r, θ, ϕ) the spatial variables in polar coordinates, T kinetic energy, V the solar wind speed, Γ defined as $(T + 2T_0)/(T + T_0)$, and T_0 the rest energy.

Based on Ito's proof, there is an exact equivalence of the FPE with the corresponding set of stochastic differential equations (SDEs).

In the literature, there are two different ways to find the corresponding SDEs. For example, in [1], by defining

$\mu = \cos\theta$, the FPE is,

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{r^2}\frac{\partial}{\partial r}\frac{\partial}{\partial r}\left(r^2\kappa_{rr}f\right) - \frac{1}{r^2}\frac{\partial}{\partial r}\left(f\frac{\partial r^2\kappa_{rr}}{\partial r}\right) \\ &+ \frac{\partial}{\partial\mu}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\kappa_{\theta\theta}}{r^2}f\right] \\ &- \frac{\partial}{\partial\mu}\left\{f\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\kappa_{\theta\theta}}{r^2}\right]\right\} \\ &+ \frac{\partial}{\partial T}\left(\frac{1}{3r^2}\frac{\partial r^2 V}{\partial r}\Gamma T f\right) \\ &- \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V f\right) \ . \end{aligned} \quad (3)$$

By setting $F = r^2 f$, the corresponding SDEs are

$$\begin{aligned} \Delta r &= \left(\frac{1}{r^2}\frac{\partial r^2\kappa_{rr}}{\partial r} + V\right)\Delta t \\ &+ \sqrt{2\kappa_{rr}\Delta t}dw_r \ , \end{aligned} \quad (4)$$

$$\begin{aligned} \Delta\mu &= \frac{\partial}{\partial\mu}\left[\frac{\left(1-\mu^2\right)\kappa_{\theta\theta}}{r^2}\right]\Delta t \\ &+ \frac{1}{r}\sqrt{2(1-\mu^2)\kappa_{\theta\theta}\Delta t}dw_\theta \ , \end{aligned} \quad (5)$$

$$\Delta T = -\frac{2VT\Gamma}{3r}\Delta t \ , \quad (6)$$

where dw_r, dw_θ, or dw_ϕ is a Wiener process given by the standard normal distribution.

On the contrary, in [11], by setting $F = \sin\theta r^2 f$, the corresponding SDEs are

$$\begin{aligned} \Delta r &= \left(\frac{1}{r^2}\frac{\partial r^2\kappa_{rr}}{\partial r} + V\right)\Delta t \\ &+ \sqrt{2\kappa_{rr}\Delta t}dw_r \ , \end{aligned} \quad (7)$$

$$\begin{aligned} \Delta\mu &= \left[\frac{\partial}{\partial\mu}\frac{\left(1-\mu^2\right)\kappa_{\theta\theta}}{r^2} + \mu\frac{\kappa_{\theta\theta}}{r^2}\right]\Delta t \\ &+ \frac{1}{r}\sqrt{2\kappa_{\theta\theta}\Delta t}dw_\mu \ , \end{aligned} \quad (8)$$

$$\Delta T = -\frac{1}{3r^2}\frac{\partial r^2 V}{\partial r}T\Gamma\Delta t \ . \quad (9)$$

So we have two different sets of SDEs, according to different transformations from f to F. However, it turns out that these two approaches give the same results. The results are shown in Figure (1). The solid line is the local interstellar spectra (LIS). We take it as $21.1T^{-2.8}/(1 + 5.85T^{-1.22} + 1.18T^{-2.54})(\text{sr m}^2\text{s MeV})^{-1}$ [5] where T is the kinetic energy. The same LIS is used in all other cases in this paper. The dashed line stands for the finite difference method results. The cross symbols stand for results using the first formula Equations (4) - (6) [1], and the empty circles are for results using the second formula Equations (7) - (9) [11]. Obviously, this figure shows that these two different stochastic methods both are correct. The reason for this is that the eigenvalues and the normalized eigenvectors for these SDEs are identical. Further the two stochastic approaches yield identical results to the finite difference method.

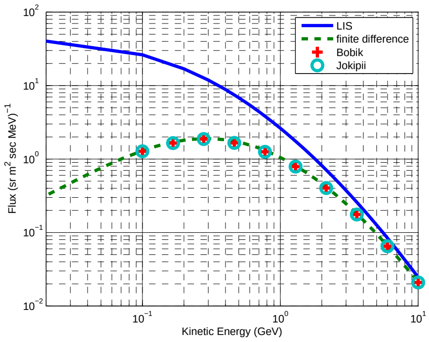

Fig. 1: 2D comparisons for the finite difference method and two different stochastic methods.

III. 3D STOCHASTIC MODEL

For 3D modulation of cosmic rays in the heliosphere, we consider two different scenarios for the cross-examination of the finite difference method and the stochastic method. The first one is 3D without drift effect and $\boldsymbol{\kappa}$ with cross-terms (such as $\kappa_{r\phi}$) suppressed. It is trivial to show that with $F = \sin\theta r^2 f$, the corresponding SDEs are

$$\begin{aligned} \Delta r &= \left(\frac{1}{r^2}\frac{\partial r^2\kappa_{rr}}{\partial r} + V\right)\Delta t \\ &+ \sqrt{2\kappa_{rr}\Delta t}dw_r \ , \end{aligned} \quad (10)$$

$$\begin{aligned} \Delta\theta &= \frac{1}{r^2\sin\theta}\frac{\partial\sin\theta\kappa_{\theta\theta}}{\partial\theta}\Delta t \\ &+ \frac{1}{r}\sqrt{2\kappa_{\theta\theta}\Delta t}dw_\theta \ , \end{aligned} \quad (11)$$

$$\begin{aligned} \Delta\phi &= \frac{1}{r^2\sin^2\theta}\frac{\partial\kappa_{\phi\phi}}{\partial\phi}\Delta t \\ &+ \frac{1}{r\sin\theta}\sqrt{2\kappa_{\phi\phi}\Delta t}dw_\phi \ , \end{aligned} \quad (12)$$

$$\Delta T = -\frac{1}{3r^2}\frac{\partial r^2 V}{\partial r}T\Gamma\Delta t \ . \quad (13)$$

Therefore we can solve these SDEs in spherical coordinates directly because the coefficient matrix in the transformed FDE only has diagonal components by the transformation from f to F in this case. Thus this matrix can be easily decomposed into two orthogonal matrices.

The comparison with the finite difference method is shown in Figure (2) and Figure (3). In Figure (2) , we use $\kappa_\parallel = \kappa_0\beta P$ where P is the particle rigidity. Constant $\kappa_0 = 2\times10^{23}\text{cm}^2\text{sec}^{-1}\text{GV}^{-1}$, and β is the ratio between particle speed and the speed of light. In Figure (3), $\kappa_\parallel = \kappa_0\beta P B_e/B$, where $\kappa_0 = 1\times10^{22}\text{cm}^2\text{sec}^{-1}\text{GV}^{-1}$, B_e is the magnetic field magnitude at Earth, and B is the magnetic field magnitude. Throughout this paper, we assume $\kappa_\perp/\kappa_\parallel = 0.1$. The LIS is denoted by the blue line. The finite difference method results are presented by the green line. And the red empty circles are our stochastic results. The agreement between these results is excellent (within 1%) for the two different diffusion coefficients.

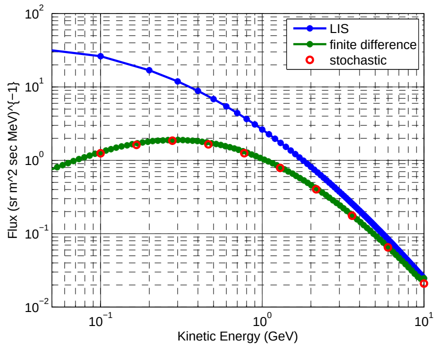

Fig. 2: 3D comparisons for the finite difference method and the stochastic method with $\kappa_{\|} = \kappa_0 \beta P$.

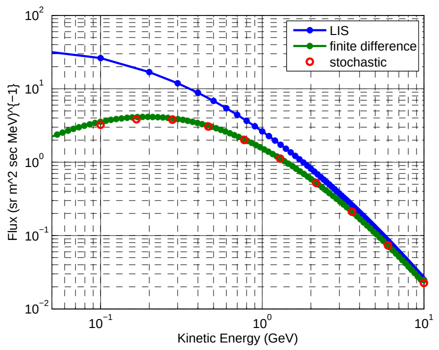

Fig. 3: 3D comparisons for the finite difference method and the stochastic method with $\kappa_{\|} = \kappa_0 \beta P B_e/B$.

The other scenario is a fully 3D model with drift effect and a fully 3D $\boldsymbol{\kappa}$ tensor including cross-terms. The corresponding SDEs are

$$\begin{aligned} \Delta r &= \left(\frac{1}{r^2}\frac{\partial r^2 \kappa_{rr}}{\partial r} + V + V_{dr}\right)\Delta t \\ &+ \sqrt{2\kappa'_{rr}\Delta t}dw_r \ , \qquad (14) \\ \Delta\theta &= \frac{1}{r^2 \sin\theta}\frac{\partial \sin\theta \kappa_{\theta\theta}}{\partial\theta}\Delta t + \frac{V_{d\theta}}{r}\Delta t \\ &+ \sqrt{2\kappa'_{\theta\theta}\Delta t}dw_\theta \ , \qquad (15) \\ \Delta\phi &= \frac{1}{r^2 \sin\theta}\frac{\partial r\kappa_{r\phi}}{\partial r}\Delta t + \frac{V_{d\phi}}{r\sin\theta}\Delta t \\ &+ \sqrt{2\kappa'_{\phi\phi}\Delta t}dw_\phi \ , \qquad (16) \\ \Delta T &= -\frac{1}{3r^2}\frac{\partial r^2 V}{\partial r}T\Gamma\Delta t \ . \qquad (17) \end{aligned}$$

The calculation of $\boldsymbol{\kappa}'$s can be done according to the transformed FPE. Note that this transformation and the Parker magnetic field enable us to solve these SDEs in spherical coordinates directly as in 2D case avoiding the transformation between Cartesian coordinates and spherical coordinates in each time step.

The Parker magnetic field used is given by

$$\mathbf{B} = \frac{A}{r^2}\left(\hat{\mathbf{e}}_r - \frac{r\Omega_\odot}{V}\sin\theta\hat{\mathbf{e}}_\phi\right)\left[1 - 2H\left(\theta - \frac{\pi}{2}\right)\right] \ .$$

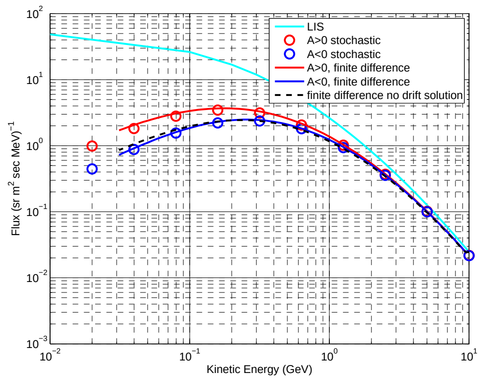

Fig. 4: 3D comparisons for the finite difference method and the stochastic method with $\kappa_{\|} = \kappa_0 \beta P B_e/B$.

The drift velocity of a particle with charge q, momentum p, and speed v is given by

$$\begin{aligned} \mathbf{V}_d &= \frac{pv}{3q}\nabla\times\left(\frac{\mathbf{B}}{B^2}\right) \\ &= \frac{2pvr}{3qA(1+\gamma^2)^2}\left[1 - 2H\left(\theta - \frac{\pi}{2}\right)\right] \\ &\times \left[-\frac{\gamma}{\tan\theta}\hat{\mathbf{e}}_r + (2+\gamma^2)\gamma\hat{\mathbf{e}}_\theta + \frac{\gamma^2}{\tan\theta}\hat{\mathbf{e}}_\phi\right] \\ &+ \frac{2pvr}{3qA(1+\gamma^2)}\delta\left(\theta - \frac{\pi}{2}\right)(\gamma\hat{\mathbf{e}}_r + \hat{\mathbf{e}}_\phi) \ , \qquad (18) \end{aligned}$$

where $A = \pm B_{0_r} R_0^2$ is a constant which comes from the definition of the magnetic field, where suffix 0 indicates some reference value, $\gamma = r\Omega_\odot \sin\theta/V$.

We adopted a formula following from a result by [4] (see also [2]) for the drift along the current sheet,

$$V_{dc} = \left(0.457 - 0.412\frac{|d|}{R_g} + 0.0915\frac{|d|^2}{R_g^2}\right)v \ , \qquad (19)$$

where d is the distance to the current sheet and R_g is the larmor radius.

Figure 4 shows the comparison between the finite difference method and the stochastic method. For A $>$ 0, the relative difference between the two is better than 10%. For A $<$ 0, the relative difference is better than 5%. We also show the case without drift which is the dashed black line. Note that spectra for A $>$ 0 are higher than for A $<$ 0. This is different from other results in the literature [7], [18], owing to the fact that we use different $\boldsymbol{\kappa}$. We have tested our model with different sets of $\boldsymbol{\kappa}$'s and find out that generally for a larger $\boldsymbol{\kappa}$ the spectrum for A $>$ 0 is higher. For a smaller $\boldsymbol{\kappa}$, the no drift spectrum is much smaller than the spectra for either A $>$ 0 or A $<$ 0.

IV. Conclusions

We described a recently developed stochastic method in great detail in this paper. We benchmarked our three-dimensional results based on the stochastic method against a well-tested three dimensional finite difference code. The excellent agreement shows that the stochastic method has a promising role in our ab initio model [16], [13] and other fully three-dimensional applications. We also show that we can use spherical coordinates for Parker's magnetic field which improves the performance of the stochastic code in 3D.

V. Acknowledgments

This work was supported in part by NASA Guest Investigator grant NNX07AH73G, NASA Heliophysics Theory grant NNX08AI47G, the Charged Sign Dependence grant NNG05WC08G, and by the South African National Research Foundation.

References

[1] Bobik, P., K. Kudela, M. Boschini, D. Grandi, M. Gervasi, and P. G. Rancoita, Solar modulation model with reentrant particles, *Advances in Space Research*, *41*, 339–342, 2008.

[2] Burger, R. A., On the Theory and Application of Drift Motion of Charged Particles in Inhomogeneous Magnetic Fields, Ph.D. thesis, Potchefstroomse Universiteit vir Christelike Hoër Onderwys, R.S.A., 1987.

[3] Burger, R. A., and M. Hattingh, Steady-State Drift-Dominated Modulation Models for Galactic Cosmic Rays, *Ap&SS*, *230*, 375–382, 1995.

[4] Burger, R. A., H. Moraal, and G. M. Webb, Drift theory of charged particles in electric and magnetic fields, *Ap&SS*, *116*, 107–129, 1985.

[5] Caballero-Lopez, R. A., H. Moraal, K. G. McCracken, and F. B. McDonald, The heliospheric magnetic field from 850 to 2000 AD inferred from ^{10}Be records, *Journal of Geophysical Research (Space Physics)*, *109*, 12,102–+, 2004.

[6] Gervasi, M., P. G. Rancoita, I. G. Usoskin, and G. A. Kovaltsov, Monte-Carlo approach to galactic cosmic ray propagation in the heliosphere., *Nuclear Physics B Proceedings Supplements*, *78*, 26–31, 1999.

[7] Jokipii, J. R., and D. A. Kopriva, Effects of particle drift on the transport of cosmic rays. III - Numerical models of galactic cosmic-ray modulation, *Astrophys. J.*, , *234*, 384–392, 1979.

[8] Jokipii, J. R., and A. J. Owens, Implications of observed charge states of low-energy solar cosmic rays, *J. Geophys. Res.*, , *80*, 1209–1212, 1975.

[9] Jokipii, J. R., and E. N. Parker, on the Convection, Diffusion, and Adiabatic Deceleration of Cosmic Rays in the Solar Wind, *Astrophys. J.*, , *160*, 735–+, 1970.

[10] Jokipii, J. R., and B. Thomas, Effects of drift on the transport of cosmic rays. IV - Modulation by a wavy interplanetary current sheet, *Astrophys. J.*, , *243*, 1115–1122, 1981.

[11] Jokipii, J. R., E. H. Levy, and W. B. Hubbard, Effects of particle drift on cosmic-ray transport. I - General properties, application to solar modulation, *Astrophys. J.*, , *213*, 861–868, 1977.

[12] Kóta, J., and J. R. Jokipii, Effects of drift on the transport of cosmic rays. VI - A three-dimensional model including diffusion, *Astrophys. J.*, , *265*, 573–581, 1983.

[13] Minnie, J., An ab initio approach to the heliospheric modulation of Galactic cosmic rays, Ph.D. thesis, North-West University, Potchefstroom, South Africa, 2006.

[14] Miyake, S., and et al., Effects of the tilted and wavy current sheet on the solar modulation of galactic cosmic rays, in *International Cosmic Ray Conference*, *International Cosmic Ray Conference*, vol. 2, pp. 203–+, 2005.

[15] Parker, E. N., The passage of energetic charged particles through interplanetary space, *Planetary & Space Science*, *13*, 9–49, 1965.

[16] Parhi, S., J. W. Bieber, W. H. Matthaeus, and R. A. Burger, Toward an Ab Initio Theory of the Solar Modulation of Cosmic Rays, *Astrophys. J.*, , *585*, 502–515, 2003.

[17] Yamada, Y., S. Yanagita, and T. Yoshida, A stochastic view of the solar modulation phenomena of cosmic rays, *Geophys. Res. Lett.*, , *25*, 2353–2356, 1998.

[18] Zhang, M., A Markov Stochastic Process Theory of Cosmic-Ray Modulation, *Astrophys. J.*, , *513*, 409–420, 1999.