Validity range of the escape term approximation in the momentum diffusion equation

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***Abstract*. We compare analytic stationary solutions of two CR transport equations: a diffusion equation with the spatial diffusion and the momentum diffusion terms and an approximate equation with the escape term, f/τ, substituted instead of the spatial diffusion term. By comparing these solutions for a 1-D slab diffusive volume we discuss physical conditions required for the approximate equation to reproduce actual particle distributions. In particular the approximate equation better reproduces distributions integrated over the source volume, but in some cases can be significantly discrepant from a varying in space local distribution.**

***Keywords*: stochastic acceleration, theory of acceleration, kinetic equation**

I. INTRODUCTION

The isotropic part of the distribution function of relativistic particles interacting with non-relativistic MHD waves obeys the diffusion-convection equation. Here we concentrate on stationary solutions of two phase-space diffusion equations: first with the spatial and momentum diffusion terms and second with the escape term, approximating effects of the spatial diffusion term. We compare both solutions in order to investigate the validity range of this approximaton. In the particular case presented in this paper we assume that particles are injected in the selected point x_0 of the accleration region with a power-law momentum distribution function, hence a source function takes form

$$Q(x,p) \sim \delta(x-x_0)\, p^{-\alpha} H(p_{max}-p) H(p-p_{min}). \quad (1)$$

For $\alpha = 4$ this kind of momentum distribution could be generated in the first order Fermi mechanism at the shock.

We consider the 1-D turbulent region extending within $-L$, L. MHD turbulence is modelled as a superposition of Alfvén waves with a simple power-law spectrum, $W(k) \propto k^{-q}$, with $1 < q < 2$.

II. EQUATION WITH THE SPATIAL DIFFUSION TERM

The presented phase-space diffusion equation

$$\frac{\partial}{\partial x}\left(\kappa(x,p)\frac{\partial f}{\partial x}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D(x,p)\frac{\partial f}{\partial p}\right) = -Q(x,p) \quad (2)$$

is a special case of the diffusion-convection equation [1]. For the assumed diffusive volume boundary conditions take form:

$$f(-L,p) = 0 = f(L,p),$$

$$\lim_{p\to\infty} f(x,p) = 0,$$

$$f(x,p=0) < \infty.$$

Let us consider the case of diffusion coefficients independent of the spatial variable x, [1]

$$\kappa(x,p) = \kappa(p) = \kappa_0\, p^{2-q},$$

$$D(x,p) = D(p) = D_0\, p^q,$$

and the source function separable in x and p variables,

$$Q(x,p) = Q_1(x)\, Q_2(p).$$

Then equation resulting from Eq. 2

$$\frac{\partial^2 f}{\partial x^2} + \frac{1}{\kappa(p)}\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D(p)\frac{\partial f}{\partial p}\right) + Q_1(x)\frac{Q_2(p)}{\kappa(p)} = 0$$

has a general form [1]

$$\mathcal{O}_x f + \frac{1}{\kappa(p)}\mathcal{O}_p f + Q_1(x)\frac{Q_2(p)}{\kappa(p)} = 0, \quad (3)$$

where $\mathcal{O}_x$ and $\mathcal{O}_p$ are differential operators in x and p variables, respectively. $\mathcal{O}_x$ is a Sturm-Liouville-type operator with negative eigenvalues, $-\beta_n^2 < 0$, and a complete set of orthogonal eigenfunctions $h_i(x)$ [2].

Eq. 3 can be solved by separation of variables:

$$f(x,p) = \sum_n A_n(x) R_n(p) = \sum_n C_n h_n(x) R_n(p), \quad (4)$$

where C_n are constants. Without loss of generality we can write

$$Q_1(x) = \sum_n A_n(x), \quad (5)$$

as there is always a possibility of rescaling $A_n(x)$ and $R_n(p)$ such that $Q_1(x) = \sum_n \tilde{A}_n(x)$ and $f(x,p) = \sum_n \tilde{A}_n(x)\tilde{R}_n(p)$. After substituting Eq. 4 into 3 we obtain a momentum equation

$$\mathcal{O}_p R_n(p) - \beta_n^2\,\kappa(p) R_n(p) = -Q_2(p), \quad (6)$$

where we use the fact that $A_n(x)$ are linearly independent.

The eigenproblem for $\mathcal{O}_x$ gives a general solution

$$A_n(x) = c_n \cos\left(\frac{(2n+1)\pi}{2L}x\right) + d_n \sin\left(\frac{n\pi}{L}x\right) = c_n h_n^{(1)}(x) + d_n h_n^{(2)}(x).$$

Coefficients c_n and d_n can be obtained from Eq. 5 by multiplying both sides by $h_m^{(1)}(x)$ or $h_m^{(2)}(x)$ and integrating over the whole region. For the special case of $Q_1(x) = \delta(x - x_0)$ we have

$$c_m = \frac{1}{L} \cos\left(\frac{(2m+1)\pi}{2L} x_0\right) \quad (7)$$

and

$$d_m = \frac{1}{L} \sin\left(\frac{m\pi}{L} x_0\right). \quad (8)$$

Eq. 6 was considered in literature many times, e.g. [1]. For dimensionless momentum $y = p/p_0$, where $p_0 = mc$[1], we obtain two independent solutions of a homogeneous equation:

$$g_1(y) = e^{-\frac{1}{2}(y/y_{cn})^\chi} M\left(\frac{3}{2\chi}, \frac{3}{\chi}, \left(\frac{y}{y_{cn}}\right)^\chi\right),$$

$$g_2(y) = e^{-\frac{1}{2}(y/y_{cn})^\chi} U\left(\frac{3}{2\chi}, \frac{3}{\chi}, \left(\frac{y}{y_{cn}}\right)^\chi\right),$$

where $M(a, b, z)$ and $U(a, b, z)$ are confluent hypergeometric functions, $\chi = 2 - q$, $y_{cn} = y_c L^{-1} \beta_n$,

$$y_c = \frac{1}{mc}\left(\frac{\chi}{2}\sqrt{\frac{D_0}{\kappa_0}} L\right)^{1/\chi} = \left(\frac{\chi}{2}\sqrt{\frac{\tau_0}{\tau_{a,0}}}\right)^{1/\chi},$$

and $\tau_0 = L^2/\kappa(p_0)$ is a characteristic diffusion (escape) timescale for particles with momenta $p_0 = mc$, $\tau_{a,0} = p_0^2/D(p_0)$ is a characteristic acceleration timescale for $p = p_0$. We note that the parameter y_c measures the strength of stochastic acceleration [1].

A Green's function can be constructed as

$$G(y, y_0) = -\frac{y_0^{-4+\chi}}{W(g_1, g_2; y_0)} \begin{cases} g_1(y)\, g_2(y_0) & y < y_0 \\ g_1(y_0)\, g_2(y) & y > y_0 \end{cases},$$

where $W(g_1, g_2; y_0)$ is the Wronskian [3].

Solution of Eq. 6 is given by an integral of the Green's and the source functions. In the particular case of the source given by Eq. 1 one obtains

$$f_d(x, y) = q\, T_f \times \sum_{n=0}^{\infty} \left\{ c_n h_n^{(1)} \int_{y_{min}}^{y_{max}} dy_0\, y_0^{-\alpha+2}\, G_n^{(1)}(y, y_0) + d_n h_n^{(2)} \int_{y_{min}}^{y_{max}} dy_0\, y_0^{-\alpha+2}\, G_n^{(2)}(y, y_0) \right\}, \quad (9)$$

where $G_n^{(1)}$ and $G_n^{(2)}$ are the Green's functions for eigenvalues $\beta_n^{(1)} = (2n+1)\pi/2L$ and $\beta_n^{(2)} = n\,\pi/L$, respectively; c_n and d_n are given by Eq. 7 and 8.

The integral momentum distribution in the slab is obtained by integrating spatial functions over the whole region:

$$f_d(y) = \frac{4\,q\,T_f}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} c_n \times \int_{y_{min}}^{y_{max}} dy_0\, y_0^{-\alpha+2}\, G_{\lambda_1,n}(y, y_0). \quad (10)$$

[1]The choice of $p_0 = mc$ defines the momentum scale, but in fact we consider only relativistic particles with momenta $p > mc$.

III. Equation with the escape term

Eq. 2 is often approximated for the full diffusive region by using the characteristic escape timescale τ. In this case we have

$$-\frac{f}{\tau(p)} + \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D(p) \frac{\partial f}{\partial p}\right) = -Q_2(p), \quad (11)$$

where, as mentioned above, we assume the value for τ:

$$\tau(p) \equiv \frac{L^2}{\kappa(p)}.$$

Eq. 11 is solved by analogy with Eq. 6, yielding

$$f_\tau(y) = q\, T_f \int_{y_{min}}^{y_{max}} dy_0\, y_0^{-\alpha+2}\, G(y, y_0). \quad (12)$$

IV. Comparison

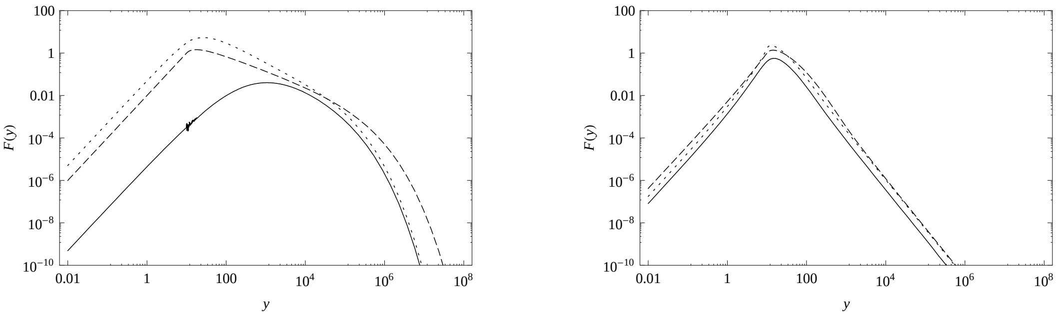
Let us compare solutions $F_d(x, y) = 4\pi y^2 f_d(x, y)$ and $F_\tau(y) = 4\pi y^2 f_\tau(y)$ of the diffusion (Eq. 2) and the approximate (Eq. 11) equations, respectively. The results for two different selected injection points x_0 are presented in Fig. 1 and 2. Here we divide functions $F_\tau(y)$ by the volume $V = 2L$ of the acceleration region in order to obtain the particle's phase-space distribution. In Fig-s 3 and 4 we show the same solutions integrated over the whole diffusive volume. In all figures functions are normalized to $F_\tau(y = 10)$ and the following parameter values are taken: $y_{min} = 10$, $y_{max} = 10^6$, $q = 1.7$. Note that all functions are smooth and visible fluctuations, e.g. Fig. 1a, have a numerical origin.

V. Conclusions

In every considered case the approximation used in Eq. 11 overestimates energies to which particles can be accelerated. Discrepancies between solutions in high-energy range mainly depend on the type of the source injecting particles, the efficiency of the stochastic acceleration y_c and the point x in which we study the particle's distribution. We note the following properties:

- for the point-injection $Q_1(x) = \delta(x - x_0)$ a change of x shifts the maximum of $F_d(x, y)$ (the nearer x_0 is the point x, the more energetic spectrum we obtain); this shift is smaller for smaller values of y_c,
- solutions $F_d(x, y)$ and $F_\tau(y)$ agree for small values of y_c - in such cases acceleration does not influence the spectrum much,
- in general, functions agree better when they are integrated over the whole region but the injection point x_0 should be sufficiently distant from the edges $-L$, L.

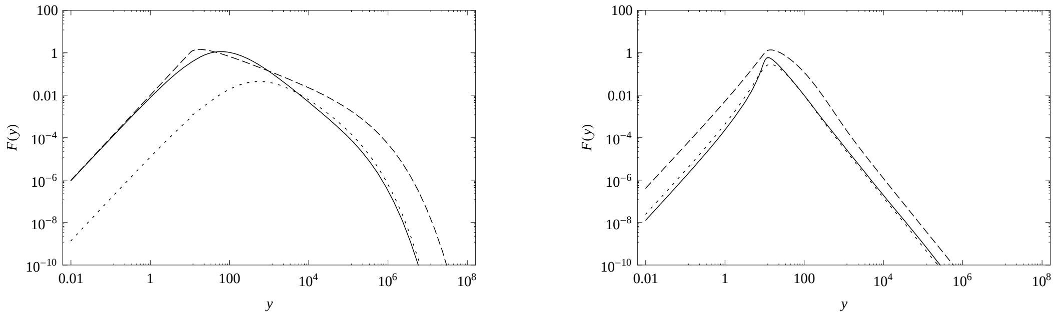
We conclude that spatial diffusion may play an important role in determining particle's distributions within spatially extended objects. In the case of the point-injection, the inclusion of diffusion effects by using the approximate escape term gives the correct phase-space distribution only when the stochastic acceleration is not effective. Calculations for other source functions will be presented elsewhere [4].



(a) $y_c = 10^2$ ($\tau_{d,0}/\tau_{a,0} \approx 704$)

(b) $y_c = 10^{-2.2}$ ($\tau_{d,0}/\tau_{a,0} \approx 2$)

Fig. 1: $x_0 = 0.01L$; $F_d(x, y)$: solid $x = 0.7L$, dotted $x = 0.1L$; $F_\tau(y)$: dashed line.



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(b) $y_c = 10^{-2.2}$ ($\tau_{d,0}/\tau_{a,0} \approx 2$)

Fig. 2: $x_0 = 0.9L$; $F_d(x, y)$: solid $x = 0.7L$, dotted $x = 0.3L$; $F_\tau(y)$: dashed line.

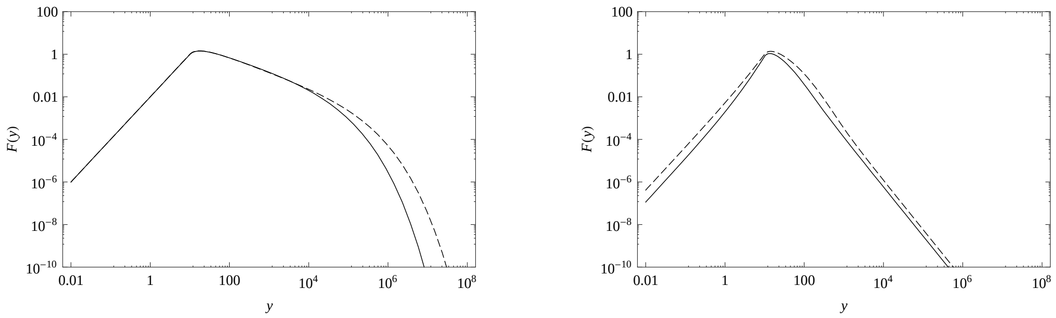

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Fig. 3: $x_0 = 0.01L$; integrated $F_d(y)$: solid; $F_\tau(y)$: dashed line.

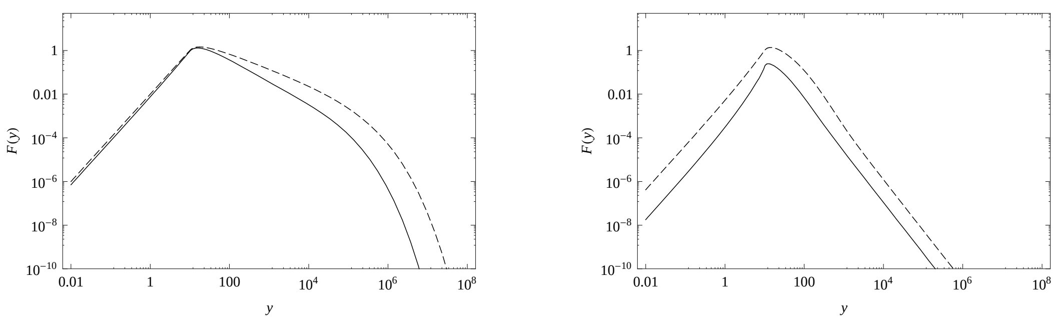

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Fig. 4: $x_0 = 0.9L$; integrated $F_d(y)$: solid; $F_\tau(y)$: dashed line.

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