Particle Diffusion Coefficients at Shock Waves

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***Abstract*. A mechanism that yields the high energies of cosmic rays is diffusive particle acceleration at shock waves. One important parameter which controls the maximum energy and the spectrum of cosmic rays is the diffusion tensor. In most previous articles about diffusive shock acceleration simple models have been used for approximating this tensor such as Bohm diffusion and isotropic scattering. Here we investigate the parameter regimes for which the assumption of Bohm diffusion and isotropic scattering are valid.**

***Keywords*: Cosmic Rays, shock waves, supernova remnants**

I. INTRODUCTION

Shocks are common in diverse astrophysical environments on scales ranging from cometary bow shocks and interplanetary shocks to supernova remnants and AGN's (active galactic nuclei). The mechanism which is responsible for the observed power-law spectra of cosmic rays is diffusive shock acceleration (Axford et al. 1977, Bell 1978a,b, Blandford & Ostriker 1978). Observed cosmic rays having energies until about 10^9eV can be explained by assuming acceleration at interplanetary shocks (Zank et al. 2000, Li et al. 2003, Li et al. 2005, Zank et al. 2006) and particles having energies until about 10^{15}eV get their high energy due to acceleration in supernova remnants (Völk et al. 1988, Lucek & Bell 2000, Berezhko & Ellison 1999, Berezhko & Völk 2007). More recently it has been shown that such acceleration processes in AGN's can lead to particle energies above 10^{20}eV (Honda & Honda 2004). Therefore, the detailed understanding of diffusive shock acceleration is important for solving one of the most important puzzles of nature, namely the origin of cosmic rays.

One part of the puzzle, which has to be solved in the theory of diffusive shock acceleration, is the interaction between charged particles and stochastic magnetic fields described by the diffusion tensor κ_{ij}. We assume that the particles are scattered by interacting with a magnetic field which is the superposition of a mean field $\vec{B}_0$ and fluctuations (wave field) $\delta\vec{B}(\vec{x})$. The mean field is usually approximated by a constant field aligned parallel to the $z-$axis ($\vec{B}_0 = B_0 \vec{e}_z$), the turbulent contribution has to be replaced by models.

The tensor κ_{ij} controls cosmic ray propagation in the solar system (see, e.g., Shalchi et al. 2006), in the interstellar medium (see, e.g., Shalchi & Schlickeiser 2005), and its knowledge is also essential for understanding the acceleration of charged cosmic rays at shock waves (see, e.g., Zank et al. 2006).

In most previous articles about diffusive shock acceleration (see, e.g., Duffy 1992, Berezhko & Völk 2007) the so-called Bohm limit was employed to replace the diffusion coefficients in the transport equation. In the Bohm limit it is assumed that the spatial diffusion coefficients in all directions (isotropic scattering is also part of the assumption) are given by (see, e.g., Berezhko & Völk 2007, Section 2)

$$\kappa^{Bohm} = \frac{vpc}{3\,|q|\,B}, \tag{1}$$

where we used the particle velocity v, the particle momentum p, the speed of light c, the particle charge q, and the magnetic field strength B. In the present paper we explore the validity of the Bohm-limit and the scattering anisotropy by using analytical theory as well as numerical methods. Details can be found in Shalchi (2009) and Shalchi & Dosch (2009).

II. PARALLEL DIFFUSION: THE BOHM-LIMIT

A. General equations

The starting point for the investigation of cosmic ray parallel scattering is the Newton-Lorentz equation whose component parallel to the mean magnetic field is

$$\dot{\mu} = \frac{\Omega}{v}\left(v_x \frac{\delta B_y}{B_0} - v_y \frac{\delta B_x}{B_0}\right). \tag{2}$$

Here we have used the pitch-angle cosine $\mu = v_{\parallel}/v$ and we have neglected electric fields since they are less important for spatial diffusion. Furthermore, we used the particle velocity v and the unperturbed gyrofrequency $\Omega = (|q|\,B_0)/(\gamma m c)$ with the Lorentzfactor γ and the particle mass m. First, we calculate the so-called pitch-angle Fokker-Planck coefficient $D_{\mu\mu}$ by employing the Kubo (1957) formula

$$D_{\mu\mu} = \int_0^{\infty} dt\ \langle \dot{\mu}(t)\dot{\mu}^*(0)\rangle\,, \tag{3}$$

where the operator $\langle \ldots \rangle$ denotes the ensemble average. The parallel spatial diffusion coefficient $\kappa_{\parallel}$ and the parallel mean free path $\lambda_{\parallel}$ are related to the pitch-angle Fokker-Planck coefficient $D_{\mu\mu}$ by (see, e.g., Earl 1974)

$$\lambda_{\parallel} = \frac{3}{v}\kappa_{\parallel} = \frac{3v}{8}\int_{-1}^{+1} d\mu\ \frac{(1-\mu^2)^2}{D_{\mu\mu}}. \tag{4}$$

By combining Eqs. (2) and (3) we notice that the ensemble average operator acts on the particle velocity components v_i as well as on the turbulent fields δB_i.

To evaluate such expressions we employ the methods of nonlinear diffusion theory (for details see Shalchi 2009). As shown there, the pitch-angle Fokker-Planck coefficient becomes

$$D_{\mu\mu} = \frac{\Omega^2(1-\mu^2)}{2B_0^2} \int d^3k \; P_{yy}(\vec{k}) \left[R_-(\vec{k}) + R_+(\vec{k}) \right] \quad (5)$$

with the resonance function

$$R_\pm(\vec{k}) = \int_0^\infty dt \; \Gamma(\vec{k}, t) e^{\pm i\Omega t} \quad (6)$$

and the characteristic function

$$\Gamma(\vec{k}, t) := \left\langle e^{i\vec{k}\cdot\vec{x}} \right\rangle = \int d^3x \; f(\vec{x}, t) e^{i\vec{k}\cdot\vec{x}} \quad (7)$$

with the particle distribution function $f(\vec{x}, t)$.

B. Analytical derivation of the Bohm limit

The correct equation for calculating the particle distribution function $f(\vec{x}, t)$ and the characteristic function $\Gamma(\vec{k}, t)$ is the Fokker-Planck equation

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f}{\partial \mu} \right). \quad (8)$$

This formula can be multiplied by $\exp(ik_\parallel z)$ and can then be integrated over z to obtain

$$\frac{\partial \Gamma}{\partial t} - ik_\parallel v\mu\Gamma = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} \right). \quad (9)$$

To solve this equation we assume that pitch-angle scattering becomes isotropic for strong turbulence

$$D_{\mu\mu}(\mu) \approx D \cdot (1-\mu^2). \quad (10)$$

In this case we have to solve Eq. (9) for $\mu = 0$ and, therefore,

$$\frac{\partial \Gamma}{\partial t} = D \left[\frac{\partial^2 \Gamma}{\partial \mu^2} - 2\mu \frac{\partial \Gamma}{\partial \mu} \right]. \quad (11)$$

This differential equation can be solved as shown in Shalchi (2006) and we find for the characteristic function

$$\Gamma(t) = 1 + \sum_{l=1}^{\infty} (2l+1) P_l^2(0) e^{-l(l+1)Dt}, \quad (12)$$

where we have used the Legendre Polynoms $P_n(x)$. With this form the resonance function for strong turbulence (ST) defined in Eq. (6) becomes

$$R_\pm^{ST} = \pi\delta(\pm\Omega) + \sum_{l=1}^{\infty} P_l^2(0) \frac{2l+1}{l(l+1)D \pm i\Omega}. \quad (13)$$

To obtain Eq. (13) we have combined Eq. (6) and Eq. (12). Since the parameter Ω cannot be zero, we have $\delta(\pm\Omega) = 0$ and, thus, the Fokker-Planck coefficient from Eq. (5) becomes

$$D_{\mu\mu}(\mu) = \frac{\xi}{2} \Omega \frac{\delta B}{B_0} (1-\mu^2) \quad (14)$$

with the constant

$$\xi^2 = 2 \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} P_l^2(0). \quad (15)$$

Here we have employed $2\int d^3k \; P_{yy}(\vec{k}) = \delta B^2$ and assumed that $D \gg \Omega$ for strong turbulence ($\delta B \gg B_0$). We have evaluated Eq. (15) numerically to find that $\xi^2 = 0.77$ and therefore $\xi = 0.88$.

The parallel mean free path can be computed by combining Eqs. (4) and (14)

$$\lambda_\parallel = \frac{3v}{4\xi\Omega} \frac{B_0}{\delta B} \int_{-1}^{+1} d\mu \; (1-\mu^2) = \frac{v}{\xi\Omega} \frac{B_0}{\delta B}. \quad (16)$$

With $\xi^{-1} = 1.14 \approx 1$ we find

$$\kappa_\parallel = \frac{v}{3} \lambda_\parallel = \frac{pvc}{3q\delta B}. \quad (17)$$

By comparing with Eq. (1) we find for strong turbulence ($B = B_0 + \delta B \approx \delta B$) indeed Bohm diffusion.

C. Quasilinear transport versus Bohm diffusion

In this paragraph we employ a test-particle code to check our theoretical result and to explore the parameter regimes for which we indeed find Bohm diffusion. We also compare these results with the traditional gyroresonance picture of cosmic ray transport obtained by employing quasilinear theory. We compute the ratio $\lambda_\parallel / l_b$ as a function of the two independent variables $R = R_L/l_b$ (R_L is the particle Larmorradius and l_b is the turbulence bendover scale) and $B = \delta B / B_0$. In the test-particle simulations we have simulated the mean magnetic field B_0 as well as the stochastic fields δB_i by employing a slab model and a Kolmogorov turbulence spectrum. Then we have solved the Newton-Lorentz equation for 1000 particles numerically. From the resulting particle trajectories it is a simple matter to compute the diffusion coefficients in the different directions (for details of the code, see Qin et al. 2002).

In Figs. 1 and 2 the diffusion coefficients from the simulations are compared with quasilinear results and the Bohm limit. In Fig. 1 we have shown the results for $R = R_L/l_b = 1.0$ for different values of the ratio $\delta B / B_0$ and in Fig. 2 the results for $R = 0.1$ corresponding to lower particle energies. In both cases the simulations approach asymptotically the Bohm-limit.

III. Perpendicular Diffusion: Scattering Anisotropy

We start our investigation by assuming that field line wandering is the most important mechanism leading to particle scattering in the direction perpendicular to the mean magnetic field $\vec{B}_0 = B_0 \vec{e}_z$. Therefore, we use as equation of motion $v_i(t) = v_z(t) B_i(\vec{x}(t)) / B_z(\vec{x}(t))$ with $i = x, y$. This formula can be derived from the field line equation $dx_i = dz B_i / B_z$. $v_i(t)$ is one of the two perpendicular components of the particle velocity, $v_z(t)$ is the particle velocity along the mean magnetic field, and $B_i(\vec{x}(t))$ is the magnetic field at the position of the charged particle $\vec{x}(t)$. By using $B_z = B_0 + \delta B_z$ and $B_i = \delta B_i$, we can derive

$$v_i(t) \left[B_0 + \delta B_z(\vec{x}(t)) \right] = \delta B_i(\vec{x}(t)) v_z(t). \quad (18)$$

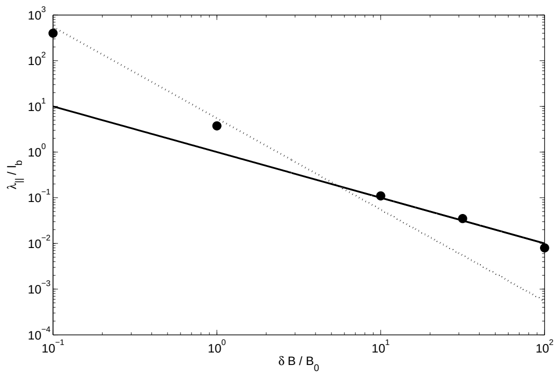

Fig. 1. Numerical test particle simulations (dots) in comparison with quasilinear results (dotted line) and the Bohm limit (solid line). The results shown here are for an intermediate particle energy $R = R_L/l_b = 1.0$. We find that quasilinear theory agrees very well with the simulations for weak and intermediate turbulence ($\delta B \leq B_0$). The Bohm limit agrees with the simulations for $\delta B > B_0$.

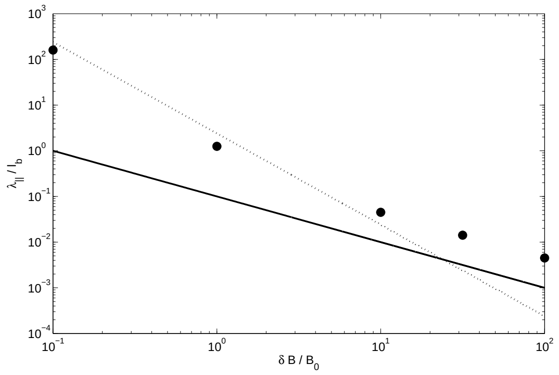

Fig. 2. Numerical test particle simulations (dots) in comparison with quasilinear results (dotted line) and the Bohm limit (solid line). The results shown here are for a low particle energy $R = R_L/l_b = 0.1$. We find that quasilinear theory agrees very well with the simulations for weak and intermediate turbulence ($\delta B \leq B_0$). The Bohm limit agrees with the simulations for $\delta B > B_0$.

To compute the diffusion coefficient κ_{ij} we employ again the Kubo formula

$$\kappa_{ij} = \int_0^\infty dt \ \langle v_i(t) v_j(0) \rangle \, . \tag{19}$$

Calculating the velocity correlation function $V_{ij} = \langle v_i(t) v_j(0) \rangle$ by using Eq. (18), is leading to fourth-order correlation functions involving particle velocities and magnetic field fluctuation values at the particle position. By following Matthaeus et al. (2003) we assume that the particle velocities are uncorrelated with the local magnetic field vector. Thus, the fourth-order correlation $< v_i v_j \delta B_k \delta B_l >$ is replaced by a product of second-order correlations $< v_i v_j >< \delta B_k \delta B_l >$. By using this approximation we find for the perpendicular diffusion coefficients

$$\kappa_{ij} = \int_0^\infty dt \ \frac{B_{ij}(\vec{x}(t))}{B_0^2 + B_{zz}(\vec{x}(t))} V_{zz}(t). \tag{20}$$

In the limit of weak turbulence ($\delta B_i \ll B_0$), Eq. (20) corresponds to the nonlinear guiding center theory developed by Matthaeus et al. (2003). For strong ($\delta B_i \gg B_0$) and isotropic turbulence ($B_{ij} = \delta_{ij} B_{xx}$) we find isotropic scattering ($\kappa_\perp = \kappa_\|$).

In order to evaluate Eq. (20) we have to specify the velocity correlation function $V_{zz}(t)$ along the mean magnetic field as well as the correlation tensor of the stochastic magnetic fields at the position of the charged particle $B_{ij}(\vec{x}(t)) = \langle \delta B_i(\vec{x}(t)) \delta B_j(\vec{x}(0)) \rangle$. For the first function we employ (see Shalchi & Döring 2007)

$$V_{zz}(t) = \frac{v^2}{3} e^{-vt/\lambda_\|}. \tag{21}$$

In order to compute the Lagrangian magnetic autocorrelations $B_{ij}(\vec{x}(t))$ we have to specify the properties of the stochastic magnetic fields which is done in the next section.

A. A simple model for the stochastic magnetic fields

The tensor $B_{ij}(\vec{x}(t))$ describes the magnetic correlation functions at the position of the charged particle. Therefore, we can use the relation

$$B_{ij}(\vec{x}(t)) = \int d^3x \ R_{ij}(\vec{x}) f_p(\vec{x}, t) \tag{22}$$

(see, e.g., Matthaeus et al. 2003, Shalchi & Dosch 2009). In Eq. (22) we have used the magnetic correlation tensor $R_{ij}(\vec{x}) =< \delta B_i(\vec{x}) \delta B_j(\vec{0}) >$ and the particle distribution function $f_p(\vec{x}, t)$. To proceed we employ a simple Gaussian and isotropic model for the correlation function, namely

$$R_{ij}(\vec{x}) = \delta_{ij} \frac{\delta B^2}{3} e^{-r^2/l_c^2}, \tag{23}$$

where we have used the correlation length l_c of the isotropic turbulence, $r = \mid \vec{x} \mid$ is the distance between the position $\vec{x}$ and the initial position. For the particle distribution we employ the axisymmetric Gaussian model

$$f_p(\vec{x}, t) = \left(64 \pi^3 t^3 \kappa_\perp^2 \kappa_\| \right)^{-1/2} e^{-\frac{\rho^2}{4\kappa_\perp t} - \frac{z^2}{4\kappa_\| t}} \tag{24}$$

with the diffusion coefficients along and across the mean field $\kappa_\|$ and $\kappa_\perp$. By combining Eqs. (22 - 24) and by solving the Gaussian integral we derive

$$B_{ij}(\vec{x}(t)) = \delta_{ij} \frac{\delta B^2}{3} \frac{l_c^3}{(l_c^2 + 4\kappa_\perp t)\sqrt{l_c^2 + 4\kappa_\| t}}. \tag{25}$$

In the following paragraphs we use this expression for computing particle diffusion coefficients.

B. Calculation of the scattering anisotropy

Here we compute the scattering anisotropy by combining Eqs. (20), (21), and (25). Furthermore, we expect for isotropic turbulence $\kappa_\perp = \kappa_{xx} = \kappa_{yy}$, $\kappa_{xy} = \kappa_{yx} = 0$, and $R_{xx}(0) = \delta B_x^2 = R_{yy}(0) = \delta B_y^2 = \delta B_z^2 = \delta B^2/3$ leading to

$$\kappa_\perp(t) = \frac{v^2}{3} \int_0^\infty dt \ e^{-vt/\lambda_\|} \times \frac{\delta B^2 l_c^3/3}{B_0^2 (l_c^2 + 4\kappa_\perp t)\sqrt{l_c^2 + 4\kappa_\| t} + \delta B^2 l_c^3/3}. \tag{26}$$

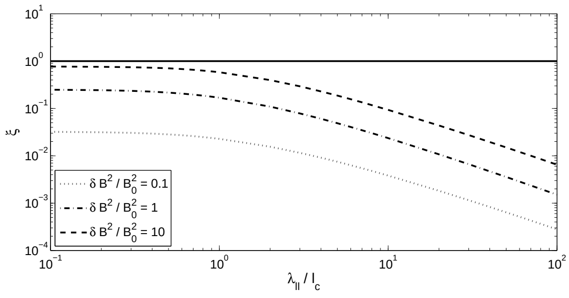

Fig. 3. The scattering anisotropy $\xi = \kappa_\perp/\kappa_\parallel$ versus the ratio $\lambda_\parallel/l_c$ for different values of $\delta B^2/B_0^2$. The solid line corresponds to isotropic scattering $\xi = 1$.

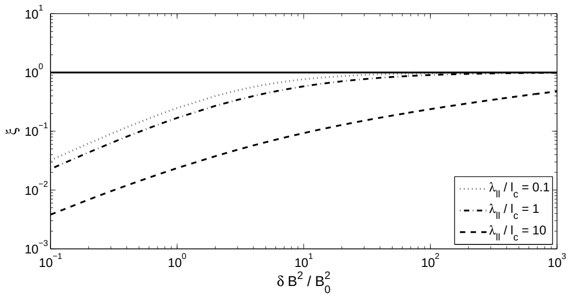

Fig. 4. The scattering anisotropy $\xi = \kappa_\perp/\kappa_\parallel$ versus the ratio $\delta B^2/B_0^2$ for different values of $\lambda_\parallel/l_c$. The solid line corresponds to isotropic scattering $\xi = 1$.

The scattering anisotropy can be defined as $\xi = \kappa_\perp/\kappa_\parallel = \lambda_\perp/\lambda_\parallel$. By employing the integral transformation $\tau = vt/\lambda_\parallel$ on Eq. (26) we find

$$\xi\left(\frac{\delta B^2}{B_0^2}, \frac{\lambda_\parallel}{l_c}\right) = \frac{\delta B^2}{3B_0^2}\int_0^\infty d\tau \; e^{-\tau} \times \left[\left(1+\frac{4}{3}\frac{\lambda_\parallel^2}{l_c^2}\xi\tau\right)\sqrt{1+\frac{4}{3}\frac{\lambda_\parallel^2}{l_c^2}\tau+\frac{\delta B^2}{3B_0^2}}\right]^{-1} \quad (27)$$

which is controlled by two parameters, namely the strength of the stochastic fields with respect to the strength of the mean field $\delta B^2/B_0^2$ and the ratio of the particle parallel mean free path and the correlation length of the turbulence $\lambda_\parallel/l_c$.

In the following we solve Eq. (27) numerically to explore the dependence of the scattering anisotropy ξ from the two parameters $\delta B^2/B_0^2$ and $\lambda_\parallel/l_c$. In Fig. 3 we illustrate the parameter ξ versus the parallel mean free path $\lambda_\parallel/l_c$ for different values of the turbulence strength $\delta B^2/B_0^2$. Fig. 4 shows the scattering anisotropy versus the turbulence strength $\delta B^2/B_0^2$ for different values of $\lambda_\parallel/l_c$. As shown by Figs. 3 and 4 we obtain a strong scattering anisotropy for the most parameter regimes. Isotropic scattering can be obtained for strong stochastic fields (with respect to the mean field) and short particle parallel mean free paths.

IV. Summary and Conclusion

By employing nonlinear diffusion theory we derived the Bohm limit analytically. In previous articles about diffusive shock acceleration this limit has been used without being justified. In the current article it is shown that the Bohm limit is indeed the correct limit for strong turbulence ($\delta B \gg B_0$) confirming the previous assumption of Bohm diffusion. We also have employed a test particle code to demonstrate that the analytical results are correct. It is shown that for weak turbulence ($\delta B \ll B_0$) traditional diffusion theory is correct whereas for strong turbulence ($\delta B \gg B_0$) we obtain Bohm diffusion.

We have also investigated the scattering anisotropy $\xi = \kappa_\perp/\kappa_\parallel$. By employing an isotropic model for the stochastic magnetic fields we derived an arbitrary formula for the parameter ξ - see Eq. (27). As shown the scattering anisotropy depends on two parameters, namely the ratios $\delta B^2/B_0^2$ and $\lambda_\parallel/l_c$. A strong scattering anisotropy corresponding to low values of ξ can be obtained for:

1) Weak stochastic magnetic fields ($\delta B^2/B_0^2 \ll 1$).
2) Long parallel mean free paths ($\lambda_\parallel \gg l_c$).

We have demonstrated systematically that we only obtain isotropic scattering for very strong stochastic fields ($\delta B \gg B_0$) or very low energetic particles ($\lambda_\parallel \ll l_c$). These results indicate that for intermediate turbulence ($\delta B \sim B_0$) which can be found in the most physical systems (interplanetary space, interstallar medium) particle scattering is highly anisotropic.

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