Particle energetic spectra induced by helical MHD turbulence

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***Abstract*. The energetic spectra of accelerated energetic particles in a helical magnetohydrodynamic turbulence are considered. The steady state cosmic ray energetic spectra as well as the solutions of the time-dependent transport equation describing stochastic particle acceleration are obtained. The evolution of accelerated particle spectra by their approach to equilibrium state is studied. Obtained results can be useful for description of particle acceleration in solar flares, supernova remnants, galactic nuclei and other astrophysical environments.**

***Keywords*: statistical acceleration, alpha-effect, energetic spectrum evolution**

I. INTRODUCTION

Here we considerate the particle acceleration in helical MHD turbulence in which the known α-effect arise and it can possess additional (initial) acceleration for more intensive mechanism [1], [2]. In fact, the plasma velocity fluctuations $\boldsymbol{u}_1$ together with the magnetic field fluctuations $\boldsymbol{H}_1$ produce a correlation $\langle[\boldsymbol{u}_1, \boldsymbol{H}_1]\rangle/c \equiv \mathcal{E}$ (the turbulent electromotive force) which allows the system to evolve back toward a stationary state. Beside it the helicity is a source of the electric field

$$\boldsymbol{E}_{(\alpha)} = -\frac{\alpha}{c}\boldsymbol{H}_0 \qquad (1)$$

along the created homogeneous MF $\boldsymbol{H}_0$ [3], [4] (α is known as the dynamo coefficient). In result, the electric field $\boldsymbol{E}_{(\alpha)}$ can accelerate the charged particles.

The momentum diffusion coefficient D_p in the transport equation [5], [6] can be written as a sum [2]

$$D_p = D_F + D_K\,, \qquad (2)$$

where

$$D_F = \frac{p^2\langle u_1^2\rangle}{3v\Lambda}\,, \quad D_K = \alpha^2\,\frac{p^2\Lambda}{3vR_H^2}\,. \qquad (3)$$

The first term D_F describes the statistical Fermi acceleration due to energetic particle scattering on moving magnetic irregularities. The second term D_K defines particle acceleration by the large-scale electric field $\boldsymbol{E}_{(\alpha)}$ arising in the turbulent medium due to α-effect. Here $R_H = pc/eH_0$ is the proton Larmour radius.

Let the particle mean free path has power law dependence on momentum,

$$\Lambda = \Lambda_0\left(\frac{\zeta}{\zeta_0}\right)^{\lambda}\,, \qquad \zeta = \frac{p}{mc}\,, \qquad (4)$$

where ζ defines the dimensionless particle momentum and m is the proton rest mass. Then the momentum diffusion coefficient (2) related to the Fermi stochastic acceleration can be written as

$$D_F = D_{0F}\zeta^{1-\lambda}\sqrt{1+\zeta^2}\,, \quad D_{0F} = \frac{m^2c\langle u_1^2\rangle\zeta_0^{\lambda}}{3\Lambda_0}\,. \qquad (5)$$

In the case of α-acceleration the coefficient D_K is

$$D_K = D_{0K}\zeta^{\lambda-1}\sqrt{1+\zeta^2}\,, \quad D_{0K} = \frac{m^2c\,\alpha^2\Lambda_0\zeta_0^{2-\lambda}}{3R_{0H}}\,. \qquad (6)$$

The quantites with subscript "0" correspond to the momentum p_0 of the particle injection.

II. STATIONARY ENERGETIC SPECTRA

For the mean free path defined by Eq. (4) the diffusion coefficient possesses the power law form:

$$D_p = D_0\zeta^{\gamma}\,. \qquad (7)$$

Defining dimensionless time,

$$\tau = \frac{t}{t_0}\,, \qquad t_0 = \frac{(mc)^2}{D_0}\,, \qquad (8)$$

the transport equation reads

$$\frac{\partial N}{\partial\tau} - \frac{1}{\zeta^2}\frac{\partial}{\partial\zeta}\zeta^{(2+\gamma)}\frac{\partial N}{\partial\zeta} + \frac{N}{\tau_e} = qt_0\,\frac{\delta(\zeta-\zeta_0)}{(mc)^3\zeta^2}\,. \qquad (9)$$

The quantity τ_e equals to the ratio of the escape time t_e to the acceleration time t_0 in (8),

$$\tau_e = \frac{t_e}{t_0}\,, \qquad (10)$$

which takes into account particle energetic losses. Let the escape time from the acceleration region is independent on CR energy ($t_e = const$). Such approximation has been used, for example, in [6]. Some authors took also the energetic dependence of t_e into considerations (see e.g. [7]). The equilibrium CR energetic spectrum arises provided the number of accelerated particles of given energy is equal to the number of particles leaving acceleration region. In that case the time derivative in the left hand side of Eq. (9) can be neglected what gives the stationary equation. The solutions of this equation have the form

$$N(\zeta) = \frac{qt_0}{\tilde{\gamma}(mc)^3}\exp\left[-\frac{1+\gamma}{2}\ln(\zeta\zeta_0)\right] \times K_{\nu}\left(\frac{\exp\left[\tilde{\gamma}\ln(\zeta_0)\right]}{\tilde{\gamma}\sqrt{\tau_e}}\right)I_{\nu}\left(\frac{\exp\left[\tilde{\gamma}\ln(\zeta)\right]}{\tilde{\gamma}\sqrt{\tau_e}}\right) \qquad (11)$$

for $\zeta < \zeta_0$, and

$$N(\zeta) = \frac{qt_0}{\tilde{\gamma}(mc)^3} \exp\left[-\frac{1+\gamma}{2}\ln(\zeta\zeta_0)\right] \times I_\nu\left(\frac{\exp[\tilde{\gamma}\ln\zeta_0]}{\tilde{\gamma}\sqrt{\tau_e}}\right) K_\nu\left(\frac{\exp[\tilde{\gamma}\ln\zeta]}{\tilde{\gamma}\sqrt{\tau_e}}\right) \quad (12)$$

for $\zeta > \zeta_0$. Here $\tilde{\gamma} = (2-\gamma)/2$ and $\nu = (1+\gamma)/(2-\gamma)$. From the last solution one obtain the expression for the density in high energy particle region:

$$N(\zeta) \propto \zeta^{-1-\frac{\gamma}{4}} \exp\left(-\frac{\exp[\tilde{\gamma}\ln\zeta]}{\tilde{\gamma}\sqrt{\tau_e}}\right). \quad (13)$$

So, the particle density exponentially decreases with energy. The solutions (11),(12) are valid only if $\gamma \neq 2$. In fact, when exponent γ approach to 2 the index ν increases indefinitely. This particular case need a special examination, as follows below.

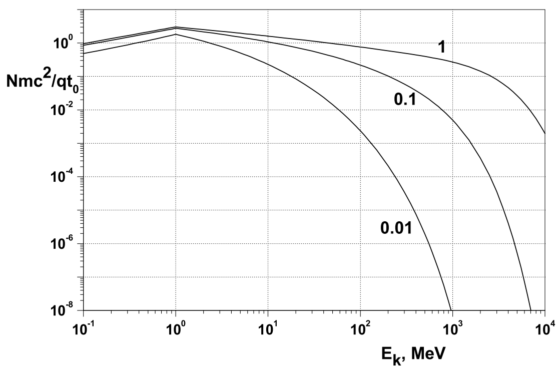

Fig. 1: The steady state energetic distribution for $\lambda = 0.5$ and $\varepsilon_{k0} = 1$ MeV. The case of Fermi acceleration.

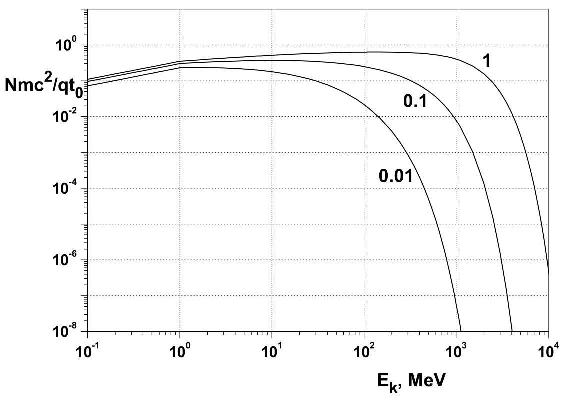

Fig. 2: As in Fig. 1 in the case of α-acceleration.

Let us consider the stochastic Fermi acceleration of particles having the constant mean free path ($\lambda = 0$). In the nonrelativistic energy region the diffusion coefficient D_p in (5) is proportional to momentum that $\gamma = 1$. Setting $\nu = 2$ in expression (12) one obtain the known expression for nonrelativistic CR density [6], [8]

$$N(\zeta) = \frac{2qt_0}{(mc)^3}\frac{1}{\zeta\zeta_0} I_2\left(2\sqrt{\frac{\zeta_0}{\tau_e}}\right) K_2\left(2\sqrt{\frac{\zeta}{\tau_e}}\right). \quad (14)$$

The momentum diffusion coefficient (5) for ultrarelativistic particles, in considered case of $\Lambda = const$, is proportional to ζ^2. Thus one has to solve the equation (9) with $\gamma = 2$. This solution takes the known form [6], [8]

$$N(\zeta) = \frac{qt_0}{2(mc)^3\sqrt{\frac{1}{\tau_e}+\frac{9}{4}}}\left(\frac{\zeta}{\zeta_0}\right)^{\Gamma_u}, \qquad \Gamma_u = -\frac{3}{2} - \sqrt{\frac{1}{\tau_e}+\frac{9}{4}} \quad (15)$$

with the power law dependence of ultrarelativistic CR density on particle momentum.

Now let us consider the steady state CR energetic spectra. If the function $N(\zeta)$ is known, one can make the change of variable ζ to kinetic energy using relationship

$$N(\varepsilon_k) = \frac{p^2}{v}N(p) = \frac{(mc)^2\zeta^2}{v}N(\zeta), \quad (16)$$

where v is the particle velocity and the dimensionless momentum is associated with kinetic energy according to

$$\zeta = \frac{\sqrt{\varepsilon_k(\varepsilon_k + 2mc^2)}}{mc^2}. \quad (17)$$

The CR steady state energetic spectra calculated according to (11),(12),(16) is shown in Fig. 1. These spectra correspond to statistical Fermi acceleration of particles with the $\Lambda \propto \sqrt{p}$ ($\lambda = 0.5$). Numbers near curves denote value τ_e of the relative rate of particle escape out of the acceleration region. The injected particle kinetic energy equals to 1 MeV and the exponent γ of the diffusion coefficient (7) is 0.5. The CR spectrum in high energy region (above the injected energy) prove to be harder with the escape time increase (Fig. 1) [6].

Now let us consider the steady state energetic spectrum typical for α-acceleration. We will exploit the same momentum dependence of mean free path ($\lambda = 0.5$) as in the case of Fermi acceleration in Fig. 1. Nonrelativistic momentum diffusion coefficient D_p (6) has the form of (7) with index $\gamma = -0.5$. In Fig. 2 the dependence of normalized particle density on kinetic energy is illustrated, given by relations (11),(12),(16). Here initial proton energy is the same ($\varepsilon_k = 1$ MeV) and numbers near the curves denote τ_e. Analogically to Fig. 1 the spectrum prove to be harder for increasing escape time. In the low energy region the CR spectrum which is suitable for α-acceleration, appears to be harder then the spectrum fitted to Fermi acceleration. On the contrary the energetic distribution of high energy CR is found to be softer for particle acceleration by the large scale electric field $\boldsymbol{E}_{(\alpha)}$, Eq. (1).

III. Time dependent energetic spectra

Let us consider the evolution of particle momentum distribution starting from time - dependent equation (9). It is convenient to use the Laplace transform

$$\Phi(\zeta, s) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-s\tau}\Phi(\zeta, \tau) \qquad (18)$$

for the new distribution function

$$\Phi(\zeta, \tau) = N(\zeta, \tau)\, \exp\left(\frac{\tau}{\tau_e}\right). \qquad (19)$$

The function (18) has to be continuous at $\zeta = \zeta_0$ (ζ_0 - the dimensionless momentum of injected particles) and its derivative on ζ at the point ζ_0 has to satisfy the condition provided by the existence of particle source in (9):

$$\frac{\partial\Phi(\zeta_0 + 0, s)}{\partial\zeta} - \frac{\partial\Phi(\zeta_0 - 0, s)}{\partial\zeta} = -\frac{t_0 q \zeta_0^{-2-q}}{(mc)^3(s - \tilde{\gamma}^2/\tau_e)} \qquad (20)$$

where $\tilde{\gamma} = (2-\gamma)/2$. This condition at $\zeta = \zeta_0$ is satisfied by the solution

$$\Phi(\zeta, s) = \frac{t_0 q}{\tilde{\gamma}(mc)^3} \frac{(\zeta\zeta_0)^{-\frac{1+\gamma}{2}}}{(s - \tilde{\gamma}^2/\tau_e)} K_\nu\left(\sqrt{s}\zeta_0^{\tilde{\gamma}}\right) I_\nu\left(\sqrt{s}\zeta^{\tilde{\gamma}}\right) \qquad (21)$$

for $\zeta < \zeta_0$, and

$$\Phi(\zeta, s) = \frac{t_0 q}{\tilde{\gamma}(mc)^3} \frac{(\zeta\zeta_0)^{-\frac{1+\gamma}{2}}}{(s - \tilde{\gamma}^2/\tau_e)} K_\nu\left(\sqrt{s}\zeta^{\tilde{\gamma}}\right) I_\nu\left(\sqrt{s}\zeta_0^{\tilde{\gamma}}\right) \qquad (22)$$

for $\zeta > \zeta_0$. Parameter γ defines momentum dependence of the diffusion coefficient $D_p(\zeta)$ in (7) and the index of Bessel functions $\nu = (1 + \gamma)/(2 - \gamma)$. Inverse Laplace transform yields

$$N(\zeta, \tau) = \frac{q t_0}{(2-\gamma)(mc)^3}(\zeta\zeta_0)^{-\frac{1+\gamma}{2}} \qquad (23)$$

$$\int_0^{\tilde{\tau}} \frac{\mathrm{d}\xi}{\xi} \exp\left[-\frac{\xi}{\tilde{\gamma}^2\tau_e} - \frac{\zeta^{2-\gamma} + \zeta_0^{2-\gamma}}{4\xi}\right] I_\nu\left(\frac{(\zeta\zeta_0)^{\tilde{\gamma}}}{2\xi}\right),$$

where the upper limit of integration is $\tilde{\tau} = \tilde{\gamma}^2\tau$.

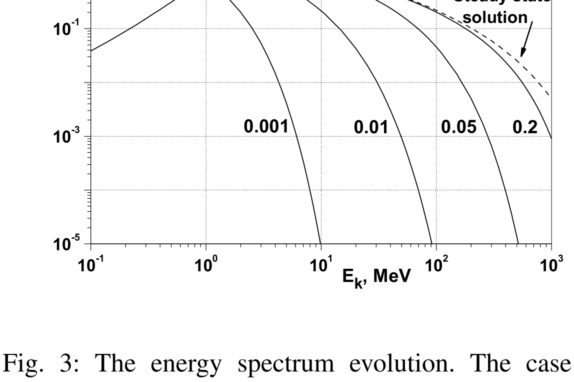

Fig. 3: The energy spectrum evolution. The case of Fermi acceleration.

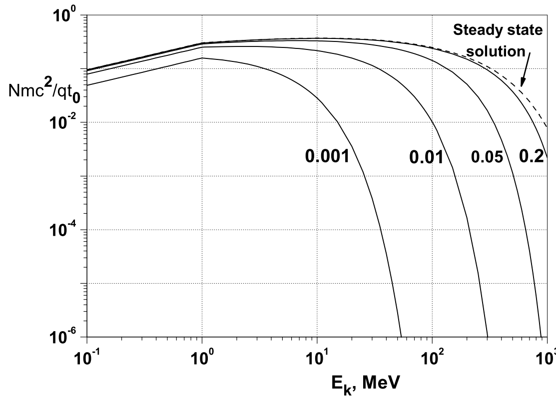

Fig. 4: As in Fig. 3 in the case of α-acceleration.

Fig. 3 shows the dependence of the normalized particle density on proton kinetic energy. The time - dependent density has been calculated using (16),(23) for following parameters: $\gamma = 0.5; \tau_e = 0.1; \varepsilon_{k0} = 1$ MeV. The value of γ corresponds to $\lambda = 1/2$ in (4) in the case of Fermi acceleration of nonrelativistic particles. Number near curves represent the dimensionless time $\tau = t/t_0$ where t_0 is given by (8). The dash curve corresponds to the steady state solution (11),(12),(16). One can see that the particle number of given energy approaches the steady state value in large time limit, therefore, the spectrum evolve gradually to this equilibrium energetic distribution.

The evolution of the spectra (under continuous particle injection) is illustrated in Fig. 4 in the case of the α-acceleration. Here $\gamma = -0.5; \tau_e = 0.1; \varepsilon_{k0} = 1$ MeV. This value of γ relates to $\lambda = 1/2$ for nonrelativistic particles accelerated by electric field (1). Numbers near solid curves are equal to τ and the dash curve represents the equilibrium spectrum. For example, in the instant $\tau = 0.05$ the spectrum is closed to the equilibrium one for $\varepsilon_k \leq 10$ MeV. At $\tau = 0.2$ the spectrum virtually match up to proton kinetic energy $\varepsilon_k \simeq 200$ MeV.

The derived expressions allow to estimate the typical time of CR spectrum approaching to the equilibrium energetic distribution. Previously the magnitude of specific acceleration time about 6 seconds has been used in (8),(5) for the initial kinetic energy $\varepsilon_{k0} = 1$ MeV and $\lambda = 0.5$. Let one choices physical properties of the acceleration region, $H_0 = 100$ gauss, $\Lambda_0 = 100\,R_0$, $u_1 = 10^8$ cm/sec [6], [9], [10], [11]. Then calculation in the case of Fermi acceleration with the diffusion coefficient (3) and the escape time $\tau_e = 0.1$ gives at instant 3 second (past their injection) the number of particles of $\varepsilon_k = 1$ GeV differing from the equilibrium value less than $\approx 5\%$. The energy of continuously injected particles was $\varepsilon_{k0} = 1$ MeV in calculation of the equilibrium spectrum. The acceleration time for protons of $\varepsilon_k = 100$ MeV will be roughly equal to one second (see in Fig. 3). The steady state CR spectrum is formed rapidly when the intensity of particle escape from the acceleration region is bigger. For example, if $\tau_e = 0.01$, the particle number of 1 GeV protons coincides with the equilibrium value (of $\approx 5.10^{-2}$) past 1 second of continuous injection of 1 MeV protons. Note that these obtained values of acceleration times sufficiently good agree with both the observational data and the known consideration of energetic particle statistical acceleration [6], [8], [9], [10].

IV. Conclusion

The solutions of transport equation describing particle acceleration are derived in stationary as well as time-dependent case. The typical acceleration times are estimated. It was shown that typical acceleration time (from proton kinetic energy $\varepsilon_{k0} = 1$ MeV up to 1 GeV) for solar coronal active regions has the order of magnitude of one or a few seconds. The obtained results can be useful for description of statistical acceleration in solar flares, supernova remnants, galactic nuclei and other astrophysical environments.

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